Announcements

1) HW #4 up on CTools

Analysis on $\mathbb{R}^n$

Definition: (norm)

Let X be a real vector space. A **norm** on X is a function $\|\cdot\| : X \to [0,\infty)$ satisfying

1) $||x|| = 0$ if and only if x is the zero vector.

2) $||cx|| = |c| \cdot ||x||$ for all $x \in \mathbb{X}$, $c \in \mathbb{R}$

3) $||x+y|| \leq ||x|| + ||y||$ (triangle inequality)

From a norm, we obtain a metric on $\mathbb{X}$,

$$d(x,y) = \|x-y\|$$

$$\forall\ x,y \in \mathbb{X}.$$

Norms on $\mathbb{R}^n$

Let $p \geq 1$, $p \in \mathbb{R}$.

Define $\|\cdot\|_p : \mathbb{R}^n \to [0, \infty)$

by, if $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$,

$$\|x\|_p = \left\{ \sum_{k=1}^{n} |x_k|^p \right\}^{1/p}$$

$$\|x\|_\infty = \max\{|x_1|, |x_2|, \ldots, |x_n|\}$$

Unless otherwise indicated, we will always take $\mathbb{R}^n$ with $\|\cdot\|_2$.

Observe: $\mathbb{R}^n$ is complete in the metric given by $\|\cdot\|_2$ (in fact, for $\|\cdot\|_p$ for any $p \geq 1$, $p = \infty$)

Extra Credit

(p-norm equivalence)

Two norms on $\mathbb{R}^n$ are **equivalent** if $\exists$ scalars $c_1, c_2 > 0$ such that for the two norms $\|\cdot\|, \|\cdot\|'$,

$$c_1 \|x\| \leq \|x\|' \leq c_2 \|x\|$$

$$\forall x \in \mathbb{R}^n.$$

Problem: Show that $\|\cdot\|_p$ is equivalent to $\|\cdot\|_2$ for any $p \geq 1$, $p = \infty$, $p \neq 2$. (Show for all n)

$$\boxed{\text{The derivative in } \mathbb{R}}$$

Let $f : \mathbb{R} \to \mathbb{R}$.

The derivative f' exists at $a \in \mathbb{R}$ if

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a).$$

Using the definition, this means:

$$\forall \ \varepsilon > 0, \ \exists \ \delta > 0 \ \ni:$$

$$\left| \frac{f(x) - f(a)}{x-a} - f'(a) \right| < \varepsilon$$

when $\quad |x-a| < \delta.$

Can rewrite one more time:

$$\left| \frac{(f(x) - f(a)) - f'(a)(x-a)}{x-a} \right| < \varepsilon$$

or, setting $x-a=h$,

$$\left| \frac{f(a+h)-f(a)-f'(a)h}{h} \right| < \varepsilon$$

Consider the function

$$g:\mathbb{R}\to\mathbb{R},$$

$$g(x)=f'(a)x.$$

The only linear maps from $\mathbb{R}$ to $\mathbb{R}$ are of this form.

We interpret the existence of the derivative as the existence of a linear function from $\mathbb{R}$ to $\mathbb{R}$ that well-approximates f "close to" a.

Extension:

Let $f: \mathbb{R}^n \to \mathbb{R}^m$.

We say f is **differentiable** at $a \in \mathbb{R}^n$ if $\exists$ a linear transformation $A: \mathbb{R}^n \to \mathbb{R}^m$ satisfying:

$\forall\ \varepsilon > 0,\ \exists\ \delta > 0$ such that:

$$\frac{\| f(a+h) - f(a) - Ah \|_2}{\| h \|_2} < \varepsilon$$

whenever $\|h\|_2 < \delta$.

We call A the derivative (or "total derivative") of f at a. We denote it by $\boxed{Df(a)}$.

Summary of Linear Transformations

Definition: A map $T: \mathbb{R}^n \to \mathbb{R}^m$ is called linear if $\forall\, x, y \in \mathbb{R}^n$ and $c \in \mathbb{R}$,

$$\boxed{T(cx+y) = cT(x) + T(y)}.$$

Recall: Any such T may be expressed as an $m \times n$ matrix via choosing a basis for $\mathbb{R}^n$. We usually choose the standard basis.

Definition: (matrix norm)

If $T: \mathbb{R}^n \to \mathbb{R}^m$ is linear, define $\|T\|$ to be the smallest (infimum) number $c \geq 0$ satisfying

$$\boxed{\|Tx\|_2 \leq c\|x\|_2}$$

for all $x \in \mathbb{R}^n$.

Note: C exists by the following observation:

$$\|Tx\|_2 \le C\|x\|_2$$ is

equivalent, by linearity of T and for $x \neq 0$, to

$$\left\| T\left(\frac{x}{\|x\|_2}\right) \right\|_2 < C.$$

What is $\left\| \frac{x}{\|x\|_2} \right\|_2$?

Since $\left\| \frac{x}{\|x\|_2} \right\|_2 = 1$

$\forall$ $x \neq 0$, we may

redefine c as

$$c = \sup_{\substack{x \in \mathbb{R}^n \\ \|x\|_2 = 1}} \|Tx\|_2 .$$

Pick x, $\|x\|_2 = 1$.

Write $x = \sum_{i=1}^{n} x_i e_i$

where $\{e_i\}_{i=1}^{n}$ is the standard basis for $\mathbb{R}^n$ and $\{x_i\}_{i=1}^{n}$ are real numbers, $\sum_{i=1}^{n} |x_i|^2 = 1$.

Then $Tx = T\left(\sum_{i=1}^{n} x_i e_i\right)$

$$= \sum_{i=1}^{n} x_i T e_i$$

Then

$$\|Tx\|_2$$

$$\leq \sum_{i=1}^{n} |x_i| \, \|Te_i\|_2$$

$$\leq \sum_{i=1}^{n} |x_i| \underbrace{\left(\max_{1 \leq i \leq n} \|Te_i\|_2 \right)}_{= s}$$

$$= s \sum_{i=1}^{n} |x_i|$$

$$\leq sn$$ since $|x_i| \leq 1 \ \forall \ 1 \leq i \leq n$.

Then

$$K = \{ \|Tx\|_2 : \|x\|_2 = 1 \}$$

is bounded above, hence admits a least upper bound which we call c!